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Left-Definite Sturm–Liouville Problems

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Left-definite regular self-adjoint Sturm–Liouville problems, with either separated or coupled boundary conditions, are studied. We give an elementary proof of the existence of eigenvalues for these problems. For any fixed equation, we establish a sequence of inequalities among the eigenvalues for different boundary conditions and estimate the range of each eigenvalue as a function on the space of boundary conditions. Some of our results here yield an algorithm for numerically computing the eigenvalues of a left-definite problem with an arbitrary coupled boundary con-

general left-definite problem on all the parameters in its differential equation and boundary condition.

Key Words: Sturm–Liouville problems; left-definiteness; existence of eigenvalues; eigenvalue inequalities; dependence of eigenvalues on parameters.

1. INTRODUCTION

In this paper, we study Sturm–Liouville problems (SLP's) associated with the regular differential equation (DE)

$$-(py')' + qy = \lambda wy \qquad \text{on} \quad J = (a, b), \tag{1.1}$$

where p, q and w satisfy the conditions

$$1/p, q, w \in \mathrm{L}^1(J, \mathbb{R}), \qquad p > 0 \text{ a.e. on } J, \qquad w \text{ changes sign on } J. \tag{1.2}$$

Here $\mathrm{L}^1(J, \mathbb{R})$ denotes the space of real-valued Lebesgue integrable functions on J. The boundary conditions (BC's) used in these problems will always be self-adjoint, see (2.4) and (2.5).

In 1918, Richardson [24] stated that even the SLP consisting of (1.1) and the Dirichlet BC can have non-real eigenvalues. However, there is an important class of SLP's associated with (1.1), the so-called "left-definite" problems, whose eigenvalues are all real. This class is studied here. There is

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an extensive literature on right-indefinite problems, including the left-definite problems; see, for example, Atkinson and Jabon [2], Atkinson and Mingarelli [3], Bennewitz and Everitt [7], Binding and Browne [4], Binding and Huang [5], Binding and Volkmer [6], Curgus and Langer [8], Daho and Langer [10], Haupt [15], Hilbert [16], Ince [17], Kamke [18], Mingarelli [23], Richardson [24] and the references cited there. Nevertheless, compared to the right-definite case, much less is known for the left-definite problems.

We first clarify the following characterization of the left-definite problems in terms of right-definite ones: the SLP consisting of the DE (1.1) and either a separated BC or a coupled BC is left-definite if and only if the lowest eigenvalue of the right-definite problem consisting of

$$-(py')' + qy = \lambda\,|w|\,y \qquad \text{on } J \tag{1.3}$$

and the same BC is positive. Actually, left-definiteness does not depend on the weight function w in (1.1), since the weight function $|w|$ in (1.3) can be replaced by any positive and integrable function on J (see also Proposition 2.6 in [8]). This characterization makes clear the dependence of left-definiteness on the coefficients p, q, the endpoints a, b, and the BC. Note, in particular, that although p is assumed to be positive, there is no sign restriction on q. We also note that the interval J of the regular Sturm–Liouville equation (1.1) is allowed to be infinite.

It has been shown (see, for example, [17]) that the eigenvalues of certain left-definite problems with *separated* BC's can be numbered by the index set

$$\mathbb{Z}^* = \{..., -2, -1, -0, 0, 1, 2, ...\} \tag{1.4}$$

such that

$$\cdots < \lambda_{-2} < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \lambda_2 < \cdots \tag{1.5}$$

and for each $n \in \mathbb{Z}^*$, the eigenfunctions (which are unique up to constant multiples) for λ_n have exactly $|n|$ zeros in the open interval (a, b). We give elementary proofs of the above facts for the general left-definite problem with a separated BC and of the following facts: any left-definite problem with a coupled BC has countably infinitely many eigenvalues, and they have neither upper bound nor lower bound and can be indexed to satisfy

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0 < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \tag{1.6}$$

with only geometrically double eigenvalues appearing twice.

According to a well-known classical result [25], the following inequalities hold in the positive right-definite case

$$\lambda_0^N \leqslant \lambda_0^P < \lambda_0^S \leqslant \{\lambda_0^D, \lambda_1^N\} \leqslant \lambda_1^S < \lambda_1^P \leqslant \{\lambda_1^D, \lambda_2^N\}$$
$$\leqslant \lambda_2^P < \lambda_2^S \leqslant \{\lambda_2^D, \lambda_3^N\} \leqslant \lambda_3^S < \lambda_3^D \leqslant \{\lambda_3^D, \lambda_4^N\} \leqslant \cdots, \tag{1.7}$$

where $\{\lambda_n^N\}$, $\{\lambda_n^P\}$, $\{\lambda_n^S\}$ and $\{\lambda_n^D\}$ denote the Neumann, periodic, semi-periodic and Dirichlet eigenvalues, respectively, and the notation $\{\lambda_0^D, \lambda_1^N\}$ with bold faced braces means each of λ_0^D and λ_1^N. In the right-definite case, these inequalities have been extended in [11] to cover an arbitrary coupled BC. Recently, analogues of the inequalities in (1.7) have been found by Constantin [9] for the left-definite case when $p = 1$.

In this paper, similar inequalities are established in the general left-definite case to cover an arbitrary coupled BC. These inequalities are comparable to those in [11] for the right-definite case, yield the above mentioned elementary proof of the existence of eigenvalues of left-definite problems with coupled BC's, and can be used to construct an algorithm for numerically computing the eigenvalues of a left-definite problem with any coupled BC. We also find the maximum and a lower bound of each positive eigenvalue as a function of the boundary condition, and obtain the minimum and an upper bound of each negative eigenvalue. Moreover, our inequalities imply that the asymptotic formula of [3] for the eigenvalues in the separated case also holds in the coupled case.

The continuous and differentiable dependence of λ_n on (all parameters of) the problem is studied. Analogues of the results recently established in [11, 19, 20, 22] for the right-definite case are found. Even though the space of left-definite problems is not open in the space of right-indefinite SLP's, each λ_n is a continuous function on the space of left-definite problems and, hence, depends continuously on each parameter, i.e., on each of a, b, $1/p$, q, w and the BC. Formulas for the derivatives, when they exist, of λ_n with respect to all parameters are found. We also give some comparison results on λ_n implied by these derivative formulas.

Our approach is based only on the basic theory of linear ordinary DE's and recent results for the right-definite case from the papers cited above. Motivated by [4, 6, 9, 17, 23], etc., we use the two-parameter equation

$$-(py')' + qy - \lambda wy = \xi\, |w|\, y \qquad \text{on } J \tag{1.8}$$

as a key feature in our approach. With this equation, the eigenvalues of the left-definite problems are connected to those of right-definite problems by "eigenvalue curves". This idea makes it possible for us to apply our recently established results for eigenvalues of right-definite problems to analyze the left-definite case.

The organization of the paper is as follows. Following this Introduction, we give in Section 2 the details on the characterization of the left-definite problems mentioned above. In Section 3, we establish inequalities among eigenvalues of left-definite problems, present some consequences of these inequalities, and comment on a generalization of left-definite problems. We study in Section 4 the dependence of the eigenvalues of the general left-definite problem on its parameters.

2. CHARACTERIZATION OF LEFT-DEFINITE PROBLEMS

We consider the SLP consisting of the DE

$$-(py')' + qy = \lambda wy \qquad \text{on} \quad J = (a, b) \tag{2.1}$$

and the self-adjoint BC

$$AY(a) + BY(b) = 0, \tag{2.2}$$

where

$$\begin{gathered} -\infty \leqslant a < b \leqslant \infty, \qquad 1/p,\, q,\, w \in \mathrm{L}^1(J, \mathbb{R}), \qquad p > 0 \text{ and } w \neq 0 \text{ a.e. on } J, \\ Y = \begin{pmatrix} y \\ y^{[1]} \end{pmatrix} \qquad \text{with} \quad y^{[1]} = py', \end{gathered} \tag{2.3}$$

while A and B are 2×2 complex-valued matrices such that

$$\text{the } 2 \times 4 \text{ matrix } (A \mid B) \text{ has full rank} \tag{2.4}$$

and

$$A \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} A^* = B \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} B^*. \tag{2.5}$$

Here A^* is the complex conjugate transpose of A.

If we abbreviate the SLP consisting of (2.1) and (2.2) as $(a, b, 1/p, q, w, A, B)$, then the space of the SLP's studied here is

$$\Omega = \{(a, b, 1/p, q, w, A, B)\colon (2.3)\text{–}(2.5) \text{ hold}\}. \tag{2.6}$$

A natural topology on Ω is the product topology induced from the usual topology on $\mathbb{R}$ and on $\mathrm{L}^1(\mathbb{R}, \mathbb{R})$. More precisely, given $\varepsilon > 0$ and $\omega_0 =$

$(a_0, b_0, 1/p_0, q_0, w_0, A_0, B_0) \in \boldsymbol{\Omega}$, the ε-neighborhood of $\boldsymbol{\omega}_0$ is defined to be the set of $\boldsymbol{\omega} = (a, b, 1/p, q, w, A, B) \in \boldsymbol{\Omega}$ satisfying

$$|a-a_0|+|b-b_0|+\int_{\mathbb{R}} (|\widetilde{1/p}-\widetilde{1/p_0}|+|\tilde{q}-\widetilde{q_0}|+|\tilde{w}-\widetilde{w_0}|) \\ +\|A-A_0\|+\|B-B_0\| < \varepsilon, \tag{2.7}$$

where $\|\cdot\|$ is any fixed matrix norm, and $\tilde{f}$ is the extension of $f \in \mathrm{L}((a, b), \mathbb{R})$ to $\mathbb{R}$ which equals 0 on $\mathbb{R}\backslash(a, b)$. It is with respect to this topology that we study the dependence of the eigenvalues of an SLP on its parameters.

The self-adjoint BC's are classified into two disjoint classes: separated and coupled. The separated self-adjoint BC's have the canonical representation

$$\begin{aligned} \cos\alpha\; y(a)+\sin\alpha\; y^{[1]}(a)=0, \\ \cos\beta\; y(b)+\sin\beta\; y^{[1]}(b)=0, \end{aligned} \tag{2.8}$$

where

$$0 < \alpha \leqslant \pi, \qquad 0 \leqslant \beta < \pi.$$

We sometimes use $(a, b, 1/p, q, w, \alpha, \beta)$ to denote the SLP consisting of (2.1) and (2.8). Each coupled self-adjoint BC can be written as

$$Y(b) = e^{i\theta} K Y(a), \tag{2.9}$$

where

$$i=\sqrt{-1}, \qquad -\pi<\theta\leqslant\pi, \qquad K\in \mathrm{SL}(2, \mathbb{R}) \tag{2.10}$$

with

$$\mathrm{SL}(2, \mathbb{R}) = \left\{ K = \begin{pmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{pmatrix} \in \mathbb{R}^{2\times 2} : \det K = 1 \right\}.$$

Let $\mathscr{D}_{\max}$ and $\mathscr{D}$ be the linear subspaces of the Hilbert space $\mathscr{H} = \mathrm{L}^2(J; |w|)$ given by

$$\begin{aligned} \mathscr{D}_{\max} &= \{f\in\mathscr{H} : f, pf'\in \mathrm{AC}_{\mathrm{loc}}(J), [-(pf')'+qf]/|w|\in\mathscr{H}\}, \\ \mathscr{D} &= \left\{ f\in\mathscr{D}_{\max} : A\begin{pmatrix} f(a) \\ pf'(a)\end{pmatrix} + B\begin{pmatrix} f(b) \\ pf'(b)\end{pmatrix} = 0 \right\}. \end{aligned} \tag{2.11}$$

Then, we define two functionals $\mathscr{R}$ and $\mathscr{L}$ on $\mathscr{D}$ as follows:

$$\mathscr{R}f = \int_a^b |f|^2 w, \qquad \mathscr{L}f = \int_a^b [-(pf')'\bar{f} + q|f|^2]. \tag{2.12}$$

Remark 2.1. In (2.2), (2.8) and (2.9), $y(a)$ and $y^{[1]}(a) = (py')(a)$ are defined by limits:

$$y(a) = \lim_{t \to a^+} y(t), \qquad (py')(a) = \lim_{t \to a^+} (py')(t). \tag{2.13}$$

By Theorem 3.4 in [27], given (2.3), these limits exist and are finite for any solution y of (2.1). Similarly, these limits exist and are finite for any maximal domain function, i.e., for any $f \in \mathscr{D}_{\max}$: let $g = -(pf')' + qf$, then

$$\int_a^b |g| = \int_a^b \frac{|g|}{|w|^{1/2}} \cdot |w|^{1/2} \leqslant \left(\int_a^b \frac{|g|^2}{|w|}\right)^{1/2} \left(\int_a^b |w|\right)^{1/2} < +\infty,$$

i.e., $g \in L^1(J)$; thus, f and pf' have finite limit at a, since f is a solution of $-(py')' + qy = g$. We have similar statements at the endpoint b. So, $\mathscr{D}$ is well-defined. Moreover, the well-definedness of $\mathscr{L}$ can be justified via the triangle inequality in $\mathscr{H}$: for any $f \in \mathscr{D}_{\max}$,

$$\begin{aligned}\int_a^b [-(pf')'\bar{f} + q|f|^2] &= \int_a^b \left[\frac{-(pf')' + qf}{|w|} \cdot \bar{f}\right] |w| \\ &\leqslant \left\| \frac{-(pf')' + qf}{|w|} \right\| \|f\| < +\infty,\end{aligned}$$

where $\|\cdot\|$ is the norm on $\mathscr{H}$.

DEFINITION 2.1. The SLP consisting of (2.1) and (2.2) is said to be *right-definite* (RD) if $\mathscr{R}$ is definite on $\mathscr{D}$, i.e., either $\mathscr{R}f > 0$ for all $f \not\equiv 0$ in $\mathscr{D}$ or $\mathscr{R}f < 0$ for all $f \not\equiv 0$ in $\mathscr{D}$; otherwise, the problem will be said to be *right-indefinite* (RID).

The SLP consisting of (2.1) and (2.2) is said to be *left-definite* (LD) if $\mathscr{L}$ is definite on $\mathscr{D}$, i.e., either $\mathscr{L}f > 0$ for all $f \not\equiv 0$ in $\mathscr{D}$ or $\mathscr{L}f < 0$ for all $f \not\equiv 0$ in $\mathscr{D}$.

It is clear that the SLP consisting of (2.1) and (2.2) is RD if and only if either $w > 0$ a.e. on J (positive RD) or $w < 0$ a.e. on J (negative RD). So, the SLP consisting of (2.1) and (2.2) is RID if and only if w changes sign on J, i.e., both sets $\{t \in J: w(t) > 0\}$ and $\{t \in J: w(t) < 0\}$ have positive

Lebesgue measure. To clarify the meaning of left-definiteness we consider the RD problem consisting of the DE

$$-(py')' + qy = \xi\, |w|\, y \qquad \text{on } J \tag{2.14}$$

and the same BC (2.2). This problem will be called the RD problem corresponding to the original SLP consisting of (2.1) and (2.2).

THEOREM 2.1. *The following three statements are equivalent:*

(i) *the Sturm–Liouville problem consisting of* (2.1) *and* (2.2) *is left-definite;*

(ii) *the functional* $\mathscr{L}$ *is positive definite on* $\mathscr{D}$, *i.e.,* $\mathscr{L}f > 0$ *for all* $f \not\equiv 0$ *in* $\mathscr{D}$;

(iii) *the eigenvalues of the right-definite problem consisting of* (2.14) *and* (2.2) *are all positive.*

Proof. (i) $\Rightarrow$ (ii). To reach a contradiction, suppose that $\mathscr{L}f < 0$ for all $f \not\equiv 0$ in $\mathscr{D}$. Let ξ_n be the $(n+1)$-th eigenvalue of the RD problem consisting of (2.14) and (2.2) and y_n an eigenfunction for ξ_n, where $n \in \mathbb{N}_0 = \{0, 1, 2, \ldots\}$. Then, $y_n \in \mathscr{D}$ and

$$\xi_n \int_a^b |y_n|^2\, |w| = \int_a^b [-(py_n')' + qy_n]\, \bar{y}_n = \mathscr{L}y_n < 0.$$

Hence, $\xi_n < 0$ for all $n \in \mathbb{N}_0$. This contradicts the well-known fact that $\xi_n \to +\infty$ as $n \to +\infty$ in the RD case. Thus, the conclusion follows from the definition of left-definiteness.

(ii) $\Rightarrow$ (i) by definition.

(ii) $\Rightarrow$ (iii). Suppose that $\mathscr{L}f > 0$ for all $f \not\equiv 0$ in $\mathscr{D}$. Let ξ_0 be the least eigenvalue of the RD problem consisting of (2.14) and (2.2) and y_0 an eigenfunction for ξ_0. Then, as in the above, we have that $y_0 \in \mathscr{D}$ and

$$\xi_0 \int_a^b |y_0|^2\, |w| = \mathscr{L}y_0 > 0.$$

Hence, $\xi_0 > 0$.

(iii) $\Rightarrow$ (ii). Suppose that $\xi_0 > 0$. From the variational characterization of the least eigenvalue in the RD case we have that

$$\inf \frac{\mathscr{L}f}{\int_a^b |f|^2\, |w|} = \xi_0 > 0, \tag{2.15}$$

where the infimum is taken over all $f \not\equiv 0$ in $\mathscr{D}$. Hence, $\mathscr{L} f > 0$ for all $f \not\equiv 0$ in $\mathscr{D}$. ▮

The next result exhibits some subclasses of LD problems.

COROLLARY 2.1. *Assume $q \geqslant 0$ and $q \not\equiv 0$ a.e. on J. Then,*

(i) *the Sturm–Liouville problem consisting of* (2.1) *and* (2.8) *is left-definite if $\pi/2 \leqslant \alpha \leqslant \pi$ and $0 \leqslant \beta \leqslant \pi/2$;*

(ii) *the Sturm–Liouville problem consisting of* (2.1) *and* (2.9) *is left-definite if $K = \left(\begin{smallmatrix} c & 0 \\ 0 & 1/c \end{smallmatrix}\right)$ for some real number $c \neq 0$.*

Proof. For both cases, we let ξ_0 be the least eigenvalue of the corresponding RD problem and y an eigenfunction for ξ_0. Then, from (2.14) and by integration by parts we have that

$$\begin{aligned} \xi_0 \int_a^b |y|^2 \, |w| &= y^{[1]}(a) \, \bar{y}(a) - y^{[1]}(b) \, \bar{y}(b) + \int_a^b [p \, |y'|^2 + q \, |y|^2] \\ &> y^{[1]}(a) \, \bar{y}(a) - y^{[1]}(b) \, \bar{y}(b). \end{aligned} \tag{2.16}$$

(i) To prove the case for $\pi/2 < \alpha \leqslant \pi$ and $0 \leqslant \beta < \pi/2$, we may assume that y is real-valued and note that by (2.8),

$$y^{[1]}(a) \, \bar{y}(a) - y^{[1]}(b) \, \bar{y}(b) = -|y^{[1]}(a)|^2 \tan \alpha + |y^{[1]}(b)|^2 \tan \beta \geqslant 0. \tag{2.17}$$

The combination of (2.16) and (2.17) implies that $\xi_0 > 0$ and, hence, the SLP consisting of (2.1) and (2.8) is LD by Theorem 2.1. The subcases with $\alpha = \pi/2$ or $\beta = \pi/2$ can be proven similarly.

(ii) In this case we have that

$$y(b) = c e^{i\theta} y(a) \qquad \text{and} \qquad y^{[1]}(b) = \tfrac{1}{c} \, e^{i\theta} y^{[1]}(a).$$

Hence,

$$y^{[1]}(a) \, \bar{y}(a) - y^{[1]}(b) \, \bar{y}(b) = 0.$$

This together with (2.16) imply that $\xi_0 > 0$. Therefore, the SLP consisting of (2.1) and (2.9) is LD by Theorem 2.1 again. ▮

In Corollary 2.1, as in much of the literature on LD problems, it is assumed that $q \geqslant 0$. The next result shows not only that this assumption is not needed in general but also that q can even be unbounded from below.

Also, for each fixed BC and for any given p and w, there is a potential q yielding a LD problem. Moreover, Corollary 2.2 gives an explicit construction of such q's.

COROLLARY 2.2. *Denote by ξ_0 the least eigenvalue of the right-definite Sturm–Liouville problem consisting of* (2.14) *and* (2.2). *Then for any $\varepsilon > 0$, the Sturm–Liouville problem consisting of the differential equation*

$$-(py')' + [q - (\xi_0 - \varepsilon)\,|w|]\,y = \lambda w y \qquad \text{on } J \tag{2.18}$$

and the same boundary condition (2.2) *is left-definite.*

Proof. By Theorem 2.1, the SLP consisting of (2.18) and (2.2) is LD if and only if the RD problem consisting of the DE

$$-(py')' + [q - (\xi_0 - \varepsilon)\,|w|]\,y = \xi\,|w|\,y \qquad \text{on } J \tag{2.19}$$

and the BC (2.2) has only positive eigenvalues. From the definition of ξ_0 it follows that ε is the least eigenvalue of the latter. Therefore, the former is LD. ∎

Remark 2.2. When $w > 0$ a.e. on J, $L^2(J; w)$ is a Hilbert space with the inner product $(f, g) = \int_a^b f\bar{g}w$. This space is widely used to study the spectrum of the RD problems. When w changes sign on J, but $w \neq 0$ a.e. on J, $L^2(J; w)$ is a Krein space and the theory of operators in Krein spaces can be applied to study the RID problems, see [8] and [10]. In the LD case, Hilbert space operator theory has also been applied with a Hilbert space $\mathscr{H}_l$ constructed as follows: since $\mathscr{L}f$ is actually positive definite on the linear subspace $\mathscr{D}$ of $L^2(J; |w|)$, the inner product

$$\langle f, g\rangle = \int_a^b [-(pf')' + qf]\,\bar{g}$$

induces a norm $\|\cdot\|_l$ on $\mathscr{D}$. The Hilbert space $\mathscr{H}_l$ is the completion of $\mathscr{D}$ with respect to this norm.

Remark 2.3. By the following proposition, $|w|$ in (2.14) can be replaced by any positive (and integrable) weight function on J, i.e., left-definiteness does not depend on the weight function w in the DE. Hence, when J is finite, $|w|$ in (2.14) can be replaced by the constant function 1; when J is infinite, $|w|$ in (2.14) can be replaced by functions such as $1/(1+t^2)$, but not by the constant function 1.

PROPOSITION 2.1 (see also [8, Proposition 2.6]). *Consider the first eigenvalue $\lambda_0 = \lambda_0(w)$ of a right-definite and self-adjoint regular Sturm–Liouville* *problem* $(a_0, b_0, 1/p_0, q_0, w, A_0, B_0)$ *as a function of* w *on* $L^1((a_0, b_0), \mathbb{R}_+)$. *Then, either* $\lambda_0(w)$ *is always positive or* $\lambda_0(w)$ *is always non-positive.*

Proof. Assume that $\lambda_0(w_0) > 0$ for some $w_0 \in \mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$, then 0 is not an eigenvalue of the SLP $(a_0, b_0, 1/p_0, q_0, w_0, A_0, B_0)$, i.e., $\det(A_0 + B_0 \Phi(b_0, w_0, 0)) \neq 0$, where $\Phi(\cdot, w, \lambda)$ is the matrix fundamental solution satisfying $\Phi(a, w, \lambda) = I$ to the regular equation $(a_0, b_0, 1/p_0, q_0, w)$ with spectral parameter λ. Since $\Phi(b_0, w, 0)$ does not depend on $w \in \mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$, we deduce that $\det(A_0 + B_0 \Phi(b_0, w, 0))$ is independent of $w \in \mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$. Thus, $\det(A_0 + B_0 \Phi(b_0, w, 0)) \neq 0$ for any $w \in \mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$, i.e., 0 is not an eigenvalue of the SLP $(a_0, b_0, 1/p_0, q_0, w, A_0, B_0)$ for any $w \in \mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$. On the other hand, Theorem 2.1 in [20] implies that $\lambda_0(w)$ is continuous in w on $\mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$. Therefore, $\lambda_0(w)$ is always positive on $\mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$, because $\mathrm{L}^1((a_0, b_0), \mathbb{R}_+)$ is connected. ∎

3. EIGENVALUES OF LEFT-DEFINITE PROBLEMS

As for the RD case, the reality of the eigenvalues of a LD problem has an elementary proof, see, for example, [26]. Here we state the result for the convenience of the reader.

LEMMA 3.1. *If the Sturm–Liouville problem consisting of* (2.1) *and* (2.2) *is left-definite, then its eigenvalues are all real.*

The existence of eigenvalues for certain LD problems, some with separated BC's and some with coupled BC's, has also been proven (see, for example, [16–18]). In this section, we give an elementary proof of the existence of eigenvalues for the general LD problem and study the properties of these eigenvalues. For this purpose, we introduce the following DE with two parameters:

$$-(py')' + qy - \lambda w y = \xi \, |w| \, y \qquad \text{on } J, \tag{3.1}$$

in which ξ is the spectral parameter.

Remark 3.1. (i) For each fixed $\lambda \in \mathbb{R}$, the SLP consisting of (3.1) and (2.2) is RD and, hence, has a countably infinite number of eigenvalues $\{\xi_n(\lambda) : n \in \mathbb{N}_0\}$, which are all real, bounded from below and unbounded from above and can be indexed to satisfy

$$\xi_0(\lambda) \leqslant \xi_1(\lambda) \leqslant \xi_2(\lambda) \leqslant \cdots$$

with only the double eigenvalues appearing twice, and

$$\xi_n(\lambda) \to +\infty \qquad \text{as} \quad n \to +\infty.$$

The RD problem consisting of (3.1) and (2.2) will be called the two-parameter RD problem corresponding to the SLP consisting of (2.1) and (2.2).

(ii) By Theorem 2.1 in [20], for each $n \in \mathbb{N}_0$, $\xi_n(\lambda)$ is a continuous function of λ in $\mathbb{R}$.

(iii) Any given $\lambda^* \in \mathbb{R}$ is an eigenvalue of the SLP consisting of (2.1) and (2.2) if and only if $\xi_n(\lambda^*) = 0$ for some $n \in \mathbb{N}_0$, i.e., if and only if 0 is an eigenvalue of the problem consisting of (3.1) with $\lambda = \lambda^*$ and (2.2). Moreover, in this case, the two corresponding eigenspaces are equal.

(iv) By Theorem 2.1, the SLP consisting of (2.1) and (2.2) is LD if and only if $\xi_0(0) > 0$.

(v) If the SLP consisting of (2.1) and (2.2) is LD, then each of its eigenvalues is a root of $\xi_n(\lambda) = 0$ for some $n \in \mathbb{N}_0$.

LEMMA 3.2. *Let $n \in \mathbb{N}_0$ and $h \in \mathbb{R}$. Assume that $\xi_n(h) < \xi_0(0)$ and y_n is an eigenfunction for the eigenvalue $\xi_n(h)$ of the Sturm–Liouville problem consisting of* (3.1) *with $\lambda = h$ and* (2.2). *Then,*

$$h \int_a^b |y_n|^2 \, w > 0. \tag{3.2}$$

Proof. Let $k = \xi_n(h)$ and set $\xi = k + \tilde{\xi}$. Then, (3.1) becomes

$$-(py')' + (q - k\,|w|)\,y - \lambda w y = \tilde{\xi}\,|w|\,y \qquad \text{on } J. \tag{3.3}$$

Hence, the eigenvalues of the RD problem consisting of (3.3) and (2.2) are $\{\tilde{\xi}_m(\lambda) = \xi_m(\lambda) - k : m \in \mathbb{N}_0\}$, and $\tilde{\xi}_n(h) = 0$. Therefore, h is an eigenvalue of the SLP consisting of the equation

$$-(py')' + (q - k\,|w|)\,y = \lambda w y \qquad \text{on } J \tag{3.4}$$

and the BC (2.2). By assumption,

$$\tilde{\xi}_0(0) = \xi_0(0) - k > 0.$$

Note that the linear subspace $\mathscr{D}$ defined for (2.14) is equal to the corresponding linear subspace for (3.3) and the functional on $\mathscr{D}$ for (3.3) with $\lambda = 0$ corresponding to the functional $\mathscr{L}$ on $\mathscr{D}$ for (2.14) is given by

$$\mathscr{L}(k)\,f = \int_a^b [-(pf')'\,\bar{f} + (q - k\,|w|)\,|f|^2], \qquad f \in \mathscr{D}.$$

Similar to (2.15) we have for (3.3) with $\lambda = 0$ that

$$\inf \frac{\mathscr{L}(k) f}{\int_a^b |f|^2 |w|} \geqslant \tilde{\xi}_0(0) > 0, \tag{3.5}$$

where the infimum is over all $f \not\equiv 0$ in $\mathscr{D}$. Since $y_n \in \mathscr{D}$, (3.5) implies that $\mathscr{L}(k) y_n > 0$, i.e.,

$$\int_a^b [-(p y_n')' \, \bar{y}_n + (q - k |w|) |y_n|^2] > 0.$$

Note that y_n is also an eigenfunction for the eigenvalue h of the SLP consisting of (3.4) and (2.2). From the above inequality and (3.4) we obtain (3.2). ∎

LEMMA 3.3. *For $n \in \mathbb{N}_0$ and $h \in \mathbb{R}$, assume that either $\xi_n(h)$ is simple or $\xi_n(\lambda)$ is double for all λ in an open interval containing h. Then, $\xi_n(\lambda)$ is continuously differentiable at h. Furthermore, if $\xi_n(h) < \xi_0(0)$, then $h \, \xi_n'(h) < 0$.*

Proof. The continuous differentiability follows from Remark 3.1(ii) and the method for proving differentiability of continuous eigenvalue branches in [22]. To show the furthermore part, let y_n be defined as in Lemma 3.2, and set $u_n = c y_n$ for some constant $c > 0$ such that $\int_a^b |u_n|^2 |w| = 1$. From Lemma 3.2 we have

$$h \int_a^b |u_n|^2 w > 0. \tag{3.6}$$

By definition, $\xi_n(\lambda)$ is an eigenvalue of the RD problem consisting of (3.1) and (2.2). By applying the derivative formula (3.10) in [22] and the chain rule for differentiation, we obtain that

$$\xi_n'(h) = -\int_a^b |u_n|^2 w. \tag{3.7}$$

Combining (3.6) and (3.7) yields that $h \, \xi_n'(h) < 0$. ∎

COROLLARY 3.1. *For each $n \in \mathbb{N}_0$, the function $\xi = \xi_n(\lambda)$ is strictly decreasing in the region*

$$E_1 = \{(\lambda, \xi) \colon \lambda > 0, \xi < \xi_0(0)\}$$

and strictly increasing in the region

$$E_2 = \{(\lambda, \xi)\colon \lambda < 0, \xi < \xi_0(0)\}.$$

Therefore, if $\xi_n(h) < \xi_0(0)$ *for some* $h > 0$, *then* $\xi_n(\lambda)$ *is strictly decreasing for* $\lambda > h$; *if* $\xi_n(h) < \xi_0(0)$ *for some* $h < 0$, *then* $\xi_n(\lambda)$ *is strictly increasing for* $\lambda < h$.

Proof. Corollary 3.1 is not a direct consequence of Lemma 3.3 since $\xi_n(\lambda)$ does not need to be differentiable everywhere. However, the Dini derivatives of $\xi_n(\lambda)$, say $D^*\xi_n(\lambda)$, exist everywhere. By an argument similar to that given in the proof of Theorem 6.1 in [19], one can show that $D^*\xi_n(\lambda) < 0$ on E_1 and $D^*\xi_n(\lambda) > 0$ on E_2. Then, the conclusion of Corollary 3.1 follows. We omit the details of the proof. ∎

After the above discussion on eigenvalue curves, we are now ready to prove the existence of eigenvalues for an arbitrary LD problem. First, we deal with LD problems with separated BC's. We remind the reader that the notation $\mathbb{Z}^*$ is defined by (1.4).

THEOREM 3.1. *Assume that the Sturm–Liouville problem consisting of* (2.1) *and* (2.8) *is left-definite and right-indefinite. Then, all its eigenvalues are real, there exist countably infinitely many positive and negative eigenvalues, and they are unbounded from below and from above, have no finite cluster point, and can be indexed to satisfy the inequalities*

$$\cdots < \lambda_{-n} < \cdots < \lambda_{-1} < \lambda_{-0} < 0 < \lambda_0 < \lambda_1 < \cdots < \lambda_n < \cdots.$$

Moreover, for each $n \in \mathbb{Z}^*$, *any eigenfunction for* λ_n *has exactly* $|n|$ *zeros in the open interval* J.

Proof. The assumptions and Theorem 2.1 imply that in this case, the eigenvalue curves satisfy that $\xi_0(0) > 0$ and

$$\xi_0(\lambda) < \xi_1(\lambda) < \xi_2(\lambda) < \cdots, \qquad \forall \lambda \in \mathbb{R}.$$

It has been shown in [4] that

$$\lim_{\lambda \to \pm\infty} \xi_n(\lambda) = -\infty, \qquad \forall n \in \mathbb{N}_0.$$

The continuity of $\xi_n(\lambda)$ then implies that for each $n \in \mathbb{N}_0$, there exist $\lambda_{\pm n} \in \mathbb{R}$ satisfying $\lambda_{-n} < 0 < \lambda_n$ and $\xi_n(\lambda_{\pm n}) = 0$. The latter identity means that $\lambda_{\pm n}$ are eigenvalues of the SLP consisting of (2.1) and (2.8).

Next, we show that such $\lambda_{\pm n}$ are unique. If not, without loss of generality, assume that $0 < \lambda_* < \lambda^*$ are the first two consecutive positive numbers such that $\xi_n(\lambda_*) = \xi_n(\lambda^*) = 0$. Note that $\xi_n(\lambda_*) < \xi_0(0)$ since $\xi_0(0) > 0$. Thus, by Corollary 3.1, ξ_n is strictly decreasing for $\lambda \geqslant \lambda_*$. This is impossible. Therefore, ξ_n yields exactly two eigenvalues of the problem in question and the inequalities given in the theorem include all the eigenvalues of the problem. Note that the ordering of the eigenvalues is based on the inequalities in the last paragraph.

Finally, let $n \in \mathbb{Z}^*$. Then, any eigenfunction for λ_n is also an eigenfunction for $\xi_{|n|}(\lambda_n) = 0$ as an eigenvalue of the RD SLP consisting of (3.1) with $\lambda = \lambda_n$ and (2.8) and hence has exactly $|n|$ zeros in the open interval J. ∎

Next, we consider LD problems with coupled BC's. Fix $\theta \in (-\pi, \pi]$ and $K \in \mathrm{SL}(2, \mathbb{R})$. Let $\{\xi_n(\lambda): n \in \mathbb{N}_0\}$ be the eigenvalues of the RD problem consisting of (3.1) and (2.9). Denote by $\{\eta_n(\lambda): n \in \mathbb{N}_0\}$ and $\{\zeta_n(\lambda): n \in \mathbb{N}_0\}$ the eigenvalues of (3.1) together with

$$y(a) = 0, \qquad k_{22}\, y(b) - k_{12}\, y^{[1]}(b) = 0 \tag{3.8}$$

and

$$y^{[1]}(a) = 0, \qquad k_{21}\, y(b) - k_{11}\, y^{[1]}(b) = 0, \tag{3.9}$$

respectively; and by $\{\mu_n : n \in \mathbb{Z}^*\}$ and $\{\nu_n : n \in \mathbb{Z}^*\}$ the eigenvalues of (2.1) together with (3.8) and (3.9), respectively, whenever they exist. If the SLP consisting of (2.1) and (3.8) is LD and RID, then by Theorem 3.1, μ_n exists for all $n \in \mathbb{Z}^*$, and $\eta_n(\lambda) = 0$ if and only if $\lambda = \mu_{\pm n}$ (see also the last two paragraphs of the proof of the next theorem). Similar results are true for $\zeta_n(\lambda)$ and ν_n. We now show the existence of eigenvalues for the LD problems with coupled BC's.

THEOREM 3.2. *Suppose that the Sturm–Liouville problem consisting of* (2.1) *and* (2.9) *is left-definite and right-indefinite. Then,*

(i) *for each $n \in \mathbb{N}_0$, the function $\xi_n(\lambda)$ has exactly one positive root, to be denoted by λ_n, and exactly one negative root, to be given the notation λ_{-n};*

(ii) *the problem has a countably infinite number of eigenvalues, i.e., $\{\lambda_n : n \in \mathbb{Z}^*\}$, and they satisfy the inequalities*

$$\cdots \leqslant \lambda_{-n} \leqslant \cdots \leqslant \lambda_{-1} \leqslant \lambda_{-0} < 0 < \lambda_0 \leqslant \lambda_1 \leqslant \cdots \leqslant \lambda_n \leqslant \cdots \tag{3.10}$$

with only geometrically double eigenvalues appearing twice.

Proof. Without loss of generality we only consider the case where either $k_{11} > 0$ and $k_{12} \leqslant 0$ or $k_{11} \leqslant 0$ and $k_{12} < 0$.

Since the SLP consisting of (2.1) and (2.9) is LD, we have $\xi_0(0) > 0$. Applying Theorem 3.2 in [11] to (3.1) yields that $\eta_n(\lambda) \geqslant \xi_n(\lambda)$ for all $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}_0$. In particular, $\eta_0(0) \geqslant \xi_0(0) > 0$, which means that the RID problem consisting of (2.1) and (3.8) is LD. Thus, for each $n \in \mathbb{N}_0$, $\eta_n(\lambda) = 0$ if and only if $\lambda = \mu_{\pm n}$. Since $\xi_n(0) > 0$ and $\xi_n(\mu_{\pm n}) \leqslant \eta_n(\mu_{\pm n}) = 0$, the continuity of $\xi_n(\lambda)$ implies that there exist $\lambda_{\pm n} \in \mathbb{R}$ satisfying $\lambda_{-n} < 0 < \lambda_n$ and $\xi_n(\lambda_{\pm n}) = 0$. The latter identity means that $\lambda_{\pm n}$ are eigenvalues of the SLP consisting of (2.1) and (2.9). As in the proof of Theorem 3.1, we can show that such $\lambda_{\pm n}$ are unique.

Finally, the ordering of $\{\lambda_n : n \in \mathbb{Z}^*\}$ in (3.10) stems from the fact that $\xi_{n+1}(\lambda) \geqslant \xi_n(\lambda)$ for any $\lambda \in \mathbb{R}$ and $n \in \mathbb{N}_0$. For any $n \in \mathbb{N}_0$, $\lambda_n = \lambda_{n+1}$ if and only if $0 = \xi_n(\lambda_n) = \xi_{n+1}(\lambda_{n+1})$ is a double eigenvalue of the RD problem consisting of (3.1) with $\lambda = \lambda_n = \lambda_{n+1}$ and (2.9). Thus, by Remark 3.1(iii), $\lambda_n = \lambda_{n+1}$ for some $n \in \mathbb{N}_0$ if and only $\lambda_n = \lambda_{n+1}$ has geometric multiplicity 2. Moreover, in this case, $\lambda_{n-1} \neq \lambda_n$ if $n \geqslant 1$, and $\lambda_{n+1} \neq \lambda_{n+2}$, since the multiplicity of $0 = \xi_n(\lambda_n) = \xi_{n+1}(\lambda_{n+1})$ as an eigenvalue of the RD problem consisting of (3.1) with $\lambda = \lambda_n = \lambda_{n+1}$ and (2.9) cannot be $\geqslant 3$. Therefore, the positive eigenvalues in (3.10) are listed according to their geometric multiplicities. A similar statement is true for the negative eigenvalues in (3.10). ▮

Remark 3.2. Theorems 3.1 and 3.2 together imply that if the SLP consisting of (2.1) and (2.2) is LD and RID, then it has a countably infinite number of eigenvalues; the eigenvalues are all real, are unbounded from below and from above, and have no finite cluster point; and 0 is never an eigenvalue.

We use the graph on the next page to indicate the relations among the functions $\{\xi_n(\lambda) : n \in \mathbb{N}_0\}$ and the eigenvalues $\{\lambda_n : n \in \mathbb{Z}^*\}$.

We will adopt the indexing scheme given by Theorems 3.1 and 3.2 for the eigenvalues of a LD problem that is RID. It is with respect to this indexing scheme that we study the continuity of λ_n with respect to the parameters of the problem and also the relationships of the eigenvalues for different BC's.

Next, we establish some inequalities among the eigenvalues of a LD and RID problem with a coupled BC and those for two corresponding separated BC's.

THEOREM 3.3. *Fix a* $K \in SL(2, \mathbb{R})$.

(a) *Assume that* $k_{11} > 0$, $k_{12} \leqslant 0$, *and the Sturm–Liouville problem consisting of* (2.1) *and* (3.9) *is left-definite and right-indefinite. Then, both*

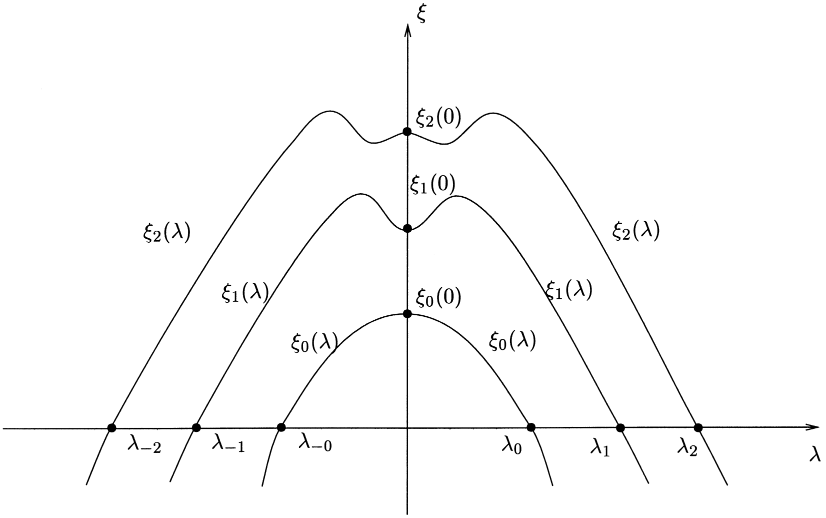

FIG. 1. The eigenvalues $\lambda_{\pm n}$ as the zeros of the functions $\xi_n(\lambda)$.

the problem consisting of (2.1) *and* (2.9) *with any* $\theta \in (-\pi, \pi]$ *and that consisting of* (2.1) *and* (3.8) *are left-definite and right-indefinite. Furthermore,* $\lambda_{\pm 0}(K)$ *are geometrically simple, and for each* $\theta \in (-\pi, \pi)$, $\theta \neq 0$, *we have*

$$\begin{aligned} \nu_0 &\leqslant \lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leqslant \{\mu_0, \nu_1\} \\ &\leqslant \lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leqslant \{\mu_1, \nu_2\} \\ &\leqslant \lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leqslant \{\mu_2, \nu_3\} \\ &\leqslant \lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leqslant \{\mu_3, \nu_4\} \leqslant \cdots \end{aligned} \tag{3.11}$$

and another set of inequalities obtained by replacing λ_n, μ_n, ν_n, $<$ *and* $\leqslant$ *in* (3.11) *by* λ_{-n}, μ_{-n}, ν_{-n}, $>$ *and* $\geqslant$, *respectively.*

(b) *Assume that* $k_{11} \leqslant 0$, $k_{12} < 0$, *and the Sturm–Liouville problem consisting of* (2.1) *and* (2.9) *with* $\theta = 0$ *is left-definite and right-indefinite. Then, the problem consisting of* (2.1) *and* (2.9) *with any other* $\theta \in (-\pi, \pi]$, *the problem consisting of* (2.1) *and* (3.8) *and the problem consisting of* (2.1) *and* (3.9) *are all left-definite and right-indefinite. Furthermore,* $\lambda_{\pm 0}(K)$ *are geometrically simple, and for each* $\theta \in (-\pi, \pi)$, $\theta \neq 0$, *we have*

$$\begin{aligned} &\lambda_0(K) < \lambda_0(e^{i\theta}K) < \lambda_0(-K) \leqslant \{\mu_0, \nu_0\} \leqslant \\ &\lambda_1(-K) < \lambda_1(e^{i\theta}K) < \lambda_1(K) \leqslant \{\mu_1, \nu_1\} \leqslant \\ &\lambda_2(K) < \lambda_2(e^{i\theta}K) < \lambda_2(-K) \leqslant \{\mu_2, \nu_2\} \leqslant \\ &\lambda_3(-K) < \lambda_3(e^{i\theta}K) < \lambda_3(K) \leqslant \{\mu_3, \nu_3\} \leqslant \cdots \end{aligned} \tag{3.12}$$

and another set of inequalities obtained by replacing λ_n, μ_n, ν_n, $<$ *and* $\leqslant$ *in* (3.12) *by* λ_{-n}, μ_{-n}, ν_{-n}, $>$ *and* $\geqslant$, *respectively.*

(c) *If neither case* (a) *nor case* (b) *applies to K, then either case* (a) *or case* (b) *applies to* $-K$.

Proof. (a) For each fixed $\lambda \in \mathbb{R}$, applying Theorem 3.2 (a) in [11] to the RD problem consisting of (3.1) and (2.9) yields that

$$\begin{aligned}\zeta_0(\lambda) &\leqslant \xi_0(\lambda, K) < \xi_0(\lambda, e^{i\theta}K) < \xi_0(\lambda, -K) \leqslant \{\eta_0(\lambda), \zeta_1(\lambda)\}\\ &\leqslant \xi_1(\lambda, -K) < \xi_1(\lambda, e^{i\theta}K) < \xi_1(\lambda, K) \leqslant \{\eta_1(\lambda), \zeta_2(\lambda)\}\\ &\leqslant \xi_2(\lambda, K) < \xi_2(\lambda, e^{i\theta}K) < \xi_2(\lambda, -K) \leqslant \{\eta_2(\lambda), \zeta_3(\lambda)\}\\ &\leqslant \xi_3(\lambda, -K) < \xi_3(\lambda, e^{i\theta}K) < \xi_3(\lambda, K) \leqslant \{\eta_4(\lambda), \zeta_4(\lambda)\} \leqslant \cdots. \end{aligned} \tag{3.13}$$

In particular, $\zeta_0(0) > 0$ implies that $\xi_0(0) > 0$ and $\eta_0(0) > 0$. Hence, by Theorem 2.1, the SLP consisting of (2.1) and (2.9) and that consisting of (2.1) and (3.8) are LD (and RID). Note that for $n \in \mathbb{N}_0$, $\lambda_{\pm n}$, $\mu_{\pm n}$ and $\nu_{\pm n}$ are the positive and negative roots of $\xi_n(\lambda)$, $\eta_n(\lambda)$ and $\zeta_n(\lambda)$, respectively. Since each of $\xi_n(\lambda)$, $\eta_n(\lambda)$ and $\zeta_n(\lambda)$ is continuous in λ and increasing in n, (3.13) implies (3.11) and the other set of similar inequalities.

Parts (b) and (c) can be proved in a similar way. ∎

Remark 3.3. It is well-known that the eigenvalues of a LD problem with a separated BC can be computed by using their characterization in terms of the Prüfer transformation. The inequalities of Theorem 3.3 can be used to construct an algorithm for computing the eigenvalues of a LD problem with an *arbitrary coupled* BC. Note that $\xi_0(0, e^{i\theta}K) > 0$ implies $\eta_0(0) > 0$. Thus, if the SLP consisting of (2.1) and (2.9) is LD and RID, then so is the SLP consisting of (2.1) and (3.8). In this case, one computes the μ_n's first, then uses the inequalities of Theorem 3.3 with the ν_n's removed to bound each eigenvalue for (2.9) between 0 or μ_n and μ_{n+1} for some $n \in \mathbb{N}_0$. The key point here is that there is exactly one eigenvalue for (2.9) in such an interval. Now, one applies a root finder to locate this one and only one root of the characteristic function in the interval. Such an algorithm is implemented in the new release of SLEIGN2 for the RD case.

THEOREM 3.4. *Assume that the Sturm–Liouville problem consisting of* (2.1) *and* (2.2) *is left-definite and right-indefinite. Then, the eigenvalues of the problem satisfy*

$$\lambda_{\pm n} \sim \pm \frac{n^2\pi^2}{\left[\int_a^b \left(\frac{w(t)}{p(t)}\right)_\pm dt\right]^2} \qquad as \quad n \to +\infty,$$

where f_- *and* f_+ *denote the positive and negative parts of a function* f, *respectively.*

Proof. When the BC in the problem is a separated one, the result has been proved in [3]; when the BC is a coupled one, the result then follows from the separated case, Remark 3.3 and the inequalities among $\lambda_{\pm n}$ and $\mu_{\pm n}$ in Theorem 3.3. ▮

In the following we denote by $\{\xi_n^D(\lambda): n \in \mathbb{N}_0\}$ the eigenvalues of the SLP consisting of the equation (3.1) and the Dirichlet BC

$$y(a) = y(b) = 0. \tag{3.14}$$

If the SLP consisting of (2.1) and (3.14) is LD and RID, we denote its eigenvalues by $\{\lambda_n^D: n \in \mathbb{Z}^*\}$.

THEOREM 3.5 (cf. [13, p. 258]). *Assume that the Sturm–Liouville problem consisting of* (2.1) *and* (2.2) *is left-definite and right-indefinite. Then, the problem consisting of* (2.1) *and* (3.14) *is left-definite and right-indefinite, and*

$$\lambda_n \in (0, \lambda_n^D],\ \lambda_{-n} \in [\lambda_{-n}^D, 0) \qquad for \quad n = 0, 1,$$
$$\lambda_n \in (\lambda_{n-2}^D, \lambda_n^D],\ \lambda_{-n} \in [\lambda_{-n}^D, \lambda_{-n+2}^D) \qquad for \quad n = 2, 3, 4, \ldots.$$

Proof. Since the SLP consisting of (2.1) and (2.2) is LD, we have that $\xi_0(0) > 0$. Applying Theorem 4.1 in [20] to (3.1) yields that

$$\xi_0^D(\lambda) \geqslant \xi_0(\lambda) \qquad \text{for} \quad \lambda \in \mathbb{R}. \tag{3.15}$$

In particular, $\xi_0^D(0) \geqslant \xi_0(0) > 0$. Thus, the SLP consisting of (2.1) and (3.14) is LD. From (3.15) we also get that $\xi_0(\lambda_0^D) \leqslant \xi_0^D(\lambda_0^D) = 0$. This shows that $\lambda_0 \in (0, \lambda_0^D]$ since λ_0 is the only positive root of $\xi_0(\lambda) = 0$. The rest can be proved similarly. ▮

To end this section, we comment on a class of RID problems which have essentially the same properties as those of the LD problems. This is the class of SLP's that can be transformed into LD problems by a translation. See also [1] and [5].

DEFINITION 3.1. Let $s \in \mathbb{R}$, and define a functional $\mathscr{L}_s$ on the linear subspace $\mathscr{D}$ of $L^2(J, |w|)$ given in (2.11) as follows:

$$\mathscr{L}_s f = \int_a^b [-(pf')' \bar{f} + (q - sw) |f|^2].$$

Then, the SLP consisting of (2.1) and (2.2) is said to be *s-left-definite* (*s*-LD) if $\mathscr{L}_s f > 0$ for all $f \not\equiv 0$ in $\mathscr{D}$.

Remark 3.4. From this definition it is easy to see the following:

(i) the SLP consisting of (2.1) and (2.2) is LD if and only if it is 0-LD;

(ii) the SLP consisting of (2.1) and (2.2) is *s*-LD for some $s \in \mathbb{R}$ if and only if the problem consisting of the equation

$$-(py')' + (q - s\,w)\,y = \lambda w y \qquad \text{on } (a, b) \tag{3.16}$$

and the BC (2.2) is LD;

(iii) every *s*-LD problem that is RID has countably infinitely many eigenvalues, they are all real, have neither upper bound nor lower bound and can be indexed to satisfy

$$\cdots \leqslant \lambda_{-2} \leqslant \lambda_{-1} \leqslant \lambda_{-0} < s < \lambda_0 \leqslant \lambda_1 \leqslant \lambda_2 \leqslant \cdots \tag{3.17}$$

with only geometrically double eigenvalues appearing twice;

(iv) with the relation between (2.1) and (3.16), we see that all the results above in this section hold for the *s*-LD problems if we replace the condition LD by *s*-LD and the number 0 by *s*.

Moreover, the next theorem gives two simple characterizations of the *s*-LD problems and its proof will be omitted.

THEOREM 3.6. *Assume that the Sturm–Liouville problem consisting of* (2.1) *and* (2.2) *is right-indefinite. Then, the following statements are equivalent:*

(i) *the problem is s-left-definite for some* $s \in \mathbb{R}$;

(ii) *the corresponding function* $\xi_0(\lambda)$ *has a positive value on* $\mathbb{R}$;

(iii) *the corresponding function* $\xi_0(\lambda)$ *has two distinct zeros in* $\mathbb{R}$.

COROLLARY 3.2. *If the Sturm–Liouville problem consisting of* (2.1) *and* (2.8) *has two eigenvalues* λ_* *and* λ^* *such that each of them has an eigenfunction without any zero on* (a, b), *then the problem is s-left-definite for any* $s \in (\lambda_*, \lambda^*)$.

Remark 3.5. There are results for the so-called semi-LD problems and s-semi-LD problems similar to the ones in this paper for LD problems and s-LD problems. We omit the details.

4. DEPENDENCE OF EIGENVALUES ON THE PROBLEM

In this section, we consider the continuous and differentiable dependence of the nth eigenvalue λ_n of the SLP consisting of (2.1) and (2.2) on the problem. We will use $\mathbf{\Omega}$ defined in (2.6) for the space of the SLP's considered, the topology on $\mathbf{\Omega}$ induced by the ε-neighborhoods given in (2.7) and the *jump set* (see [20])

$$\mathscr{J}^{\mathbb{C}} = \{(A \mid B) = (e^{i\theta}K \mid -I);\, K \in \mathrm{SL}(2, \mathbb{R}),\, k_{12} = 0,\, \theta \in [0, \pi)\}$$
$$\cup \left\{ (A \mid B) = \begin{pmatrix} a_1 & a_2 & 0 & 0 \\ 0 & 0 & b_1 & b_2 \end{pmatrix} \in \mathbb{R}^{2\times 4};\right.$$
$$\left. a_1^2 + a_2^2 \neq 0,\, b_1^2 + b_2^2 \neq 0,\, a_2 b_2 = 0 \right\}.$$

THEOREM 4.1. *If a Sturm–Liouville problem $\boldsymbol{\omega}_0 \in \mathbf{\Omega}$ is left-definite and right-indefinite and its boundary condition is not on the jump set $\mathscr{J}^{\mathbb{C}}$, then there is a neighborhood $\mathscr{N}$ of $\boldsymbol{\omega}_0$ in $\mathbf{\Omega}$ such that each problem in $\mathscr{N}$ is also left-definite and right-indefinite, and for any $n \in \mathbb{Z}^*$, $\lambda_n(\boldsymbol{\omega})$ is a continuous function of $\boldsymbol{\omega}$ on $\mathscr{N}$.*

Proof. For $\lambda \in \mathbb{R}$ and $\boldsymbol{\omega} \in \mathbf{\Omega}$, let $\{\xi_m(\lambda, \boldsymbol{\omega}) : m \in \mathbb{N}_0\}$ be the eigenvalues of the 2-parameter RD problem corresponding to $\boldsymbol{\omega}$, i.e., the SLP consisting of the corresponding equation (3.1) and the BC in $\boldsymbol{\omega}$. From the assumption, Theorem 2.1, and Theorem 2.1 in [20] we know that $\xi_0(0, \boldsymbol{\omega}_0) > 0$ and $\{\xi_m(\lambda, \boldsymbol{\omega}) : m \in \mathbb{N}_0\}$ are continuous on $\mathbb{R} \times \mathscr{N}$ for some neighborhood $\mathscr{N}$ of $\boldsymbol{\omega}_0$ in $\mathbf{\Omega}$. Thus, replacing $\mathscr{N}$ by a smaller neighborhood if necessary, we have that $\boldsymbol{\omega}$ is RID and $\xi_0(0, \boldsymbol{\omega}) > 0$, i.e., $\boldsymbol{\omega}$ is LD, for all $\boldsymbol{\omega} \in \mathscr{N}$. Since $\lambda_n(\boldsymbol{\omega})$ is the only positive or negative root of $\xi_{|n|}(\lambda, \boldsymbol{\omega})$ for $\boldsymbol{\omega} \in \mathscr{N}$, the continuity of $\xi_{|n|}(\lambda, \boldsymbol{\omega})$ on $\mathbb{R} \times \mathscr{N}$ implies that $\lambda_n(\boldsymbol{\omega})$ is a continuous function of $\boldsymbol{\omega}$ on $\mathscr{N}$. ∎

To see the importance of the assumption in Theorem 4.1 that the BC in $\boldsymbol{\omega}_0$ is not on the jump set $\mathscr{J}^{\mathbb{C}}$, we give the following example.

EXAMPLE. The SLP consisting of the equation

$$-y'' = \lambda(\operatorname{sgn} t)\, y \qquad \text{on } (-1, 1) \tag{4.1}$$

and the Dirichlet BC

$$y(-1) = 0 = y(1) \tag{4.2}$$

is LD (and RID) since the least eigenvalue of the SLP consisting of the Fourier equation

$$-y'' = \lambda y \qquad \text{on } (-1, 1) \tag{4.3}$$

and the BC (4.2) is positive. Now, in any neighborhood of the SLP consisting of (4.1) and (4.2), there is an SLP consisting of (4.1) and the BC

$$y(-1) - c y^{[1]}(-1) = 0, \qquad y(1) = 0 \tag{4.4}$$

for sufficiently small $c > 0$. However, this SLP is not LD for all sufficiently small $c > 0$ since the lowest eigenvalue of the SLP consisting of (4.3) and (4.4) approaches $-\infty$ as $c \to 0^+$, see [12] or [20].

COROLLARY 4.1. *If a Sturm–Liouville problem* $\boldsymbol{\omega} = (a, b, 1/p, q, w, A, B) \in \boldsymbol{\Omega}$ *is left-definite and right-indefinite, then it remains so under any sufficiently small change of its differential equation, and each* $\lambda_n(w)$ *depends continuously on the differential equation, i.e., on* a, b, $1/p$, q *and* w.

Proof. The proof is similar to that of Theorem 4.1, the difference is that here we use solely Theorem 2.1 in [20] in stead of both Theorem 2.1 in [20] and the concept of continuous eigenvalue branches. ▮

Remark 4.1. In Theorem 4.1 and Corollary 4.1 above, we left the following question unanswered: if the SLP consisting of (2.1) and (2.2) is LD and (2.2) is on the jump set $\mathscr{J}^{\mathbb{C}}$ given above, then for which BC's nearby (2.2) is the SLP still LD? Using the complete characterization in [20] of the discontinuity of the first eigenvalue in the RD case, we can show that the SLP consisting of (2.1) and a BC sufficiently close to (2.2) is LD if and only if the BC is on the "continuity" side of the jump set. This implies that the space of LD problems is not open in the space of RID SLP's, the part of its boundary contained in itself is in the jump set $\mathscr{J}^{\mathbb{C}}$, and for each $n \in \mathbb{Z}^*$, λ_n is continuous on the space of LD problems that are RID. Thus, Theorem 4.1 establishes the continuity of each $\lambda_n(\boldsymbol{\omega})$ at the interior points of this space. For more information about this space, see the forthcoming paper [14].

In the next theorem, for each $n \in \mathbb{N}_0$, $u_{\pm n}$ denote normalized eigenfunctions for the eigenvalues $\lambda_{\pm n}$ of the SLP consisting of (2.1) and (2.2), i.e., they satisfy

$$\int_a^b |u_{\pm n}|^2 \, w = \pm 1. \tag{4.5}$$

Such eigenfunctions exist since from Lemma 3.2 we have that for each $n \in \mathbb{Z}^*$ and any eigenfunction y_n for λ_n,

$$\lambda_n \int_a^b |y_n|^2 \, w > 0. \tag{4.6}$$

Moreover, in this theorem, the derivative λ_n' is the Frechet derivative in the appropriate Banach space. Recall that a map T from a Banach space X into a Banach space Y is differentiable at a point $x \in X$ if there exists a bounded linear map $T'\colon X \to Y$ satisfying

$$|T(x+h) - T(x) - T'(x)\, h| = o(h) \qquad \text{as} \quad h \to 0 \text{ in } X.$$

THEOREM 4.2. *Suppose that the Sturm–Liouville problem $\boldsymbol{\omega}_0 = (a_0, b_0, 1/p_0, q_0, w_0, A_0, B_0) \in \boldsymbol{\Omega}$ is left-definite and right-indefinite, and its boundary condition is not on the jump set $\mathcal{J}^{\mathbb{C}}$. Let $n \in \mathbb{N}_0$.*

(i) *Fix all components of $\boldsymbol{\omega}_0$ except a_0 and consider $\lambda_{\pm n}$ as a function of a. Assume that $\lambda_{\pm n}(a_0)$ is geometrically simple or $\lambda_{\pm n}(a)$ is geometrically double in some neighborhood of a_0. Then, $\lambda_{\pm n}(a)$ is differentiable a.e. in some neighborhood $\mathcal{N}_{a_0}$ of a_0 and*

$$\lambda'_{\pm n}(a) = \pm\left\{\frac{1}{p}(a)\, |pu'_{\pm n}|^2 (a) - |u_{\pm n}(a)|^2 \,[q(a) - \lambda_{\pm n}(a)\, w(a)]\right\}$$
$$\text{a.e. in } \mathcal{N}_{a_0}. \tag{4.7}$$

Furthermore, if p, q, w are continuous at a_0 and $p(a_0) \neq 0$, then (4.7)) holds at the point a_0.

(ii) *Fix all components of $\boldsymbol{\omega}_0$ except b_0 and consider $\lambda_{\pm n}$ as a function of b. Assume that $\lambda_{\pm n}(b_0)$ is geometrically simple or $\lambda_{\pm n}(b)$ is geometrically* *double in some neighborhood of* b_0. *Then,* $\lambda_{\pm n}(b)$ *is differentiable a.e. in some neighborhood* $\mathcal{N}_{b_0}$ *of* b_0 *and*

$$\lambda'_{\pm n}(b) = \pm\left\{-\frac{1}{p}(b)\,|pu'_{\pm n}|^2\,(b) + |u_{\pm n}(b)|^2\,[q(b) - \lambda_{\pm n}(b)\,w(b)]\right\}$$
$$\text{a.e. in } \mathcal{N}_{b_0}. \tag{4.8}$$

Furthermore, if p, q, w *are continuous at* b_0 *and* $p(b_0) \neq 0$, *then* (4.8) *holds at the point* b_0.

(iii) *Fix all components of* ω_0 *except* $1/p_0$ *and consider* $\lambda_{\pm n}$ *as a function of* $1/p$. *Assume that* $\lambda_{\pm n}(1/p_0)$ *is simple. Then,* $\lambda_{\pm n}(1/p)$ *is continuously differentiable in some neighborhood* $\mathcal{N}_{1/p_0}$ *of* $1/p_0$ *and for any* $1/p \in \mathcal{N}_{1/p_0}$,

$$\lambda'_{\pm n}(1/p)\,h = \pm\left\{-\int_{a_0}^{b_0} |pu'_{\pm n}|^2\,h\right\}, \qquad h \in L^1(J, \mathbb{R}). \tag{4.9}$$

(iv) *Fix all components of* ω_0 *except* q_0 *and consider* $\lambda_{\pm n}$ *as a function of* q. *Assume that* $\lambda_{\pm n}(q_0)$ *is simple. Then,* $\lambda_{\pm n}(q)$ *is continuously differentiable in some neighborhood* $\mathcal{N}_{q_0}$ *of* q_0 *and for any* $q \in \mathcal{N}_{q_0}$,

$$\lambda'_{\pm n}(q)\,h = \pm\left\{\int_{a_0}^{b_0} |u_{\pm n}|^2\,h\right\}, \qquad h \in L^1(J, \mathbb{R}). \tag{4.10}$$

(v) *Fix all components of* ω_0 *except* w_0 *and consider* $\lambda_{\pm n}$ *as a function of* w. *Assume that* $\lambda_{\pm n}(w_0)$ *is simple. Then,* $\lambda_{\pm n}(w)$ *is continuously differentiable in some neighborhood* $\mathcal{N}_{w_0}$ *of* w_0 *and for any* $w \in \mathcal{N}_{w_0}$,

$$\lambda'_{\pm n}(w)\,h = \pm\left\{-\lambda_{\pm n}(w)\int_{a_0}^{b_0} |u_{\pm n}|^2\,h\right\}, \quad h \in L^1(J, \mathbb{R}). \tag{4.11}$$

(vi) *Assume that the boundary condition in* ω_0 *is separable and is written as the canonical form* (2.8); *in this case,* A_0, B_0 *in* ω_0 *are replaced by* α_0, β_0. *Fix all components of* ω_0 *except* α_0 *and consider* $\lambda_{\pm n}$ *as a function of* α, *then* $\lambda_{\pm n}$ *is differentiable and*

$$\lambda'_{\pm n}(\alpha) = \pm\{-|u_{\pm n}|^2\,(a_0) - |pu'_{\pm n}|^2\,(a_0)\}; \tag{4.12}$$

fix all components of ω_0 *except* β_0 *and consider* $\lambda_{\pm n}$ *as a function of* β, *then* $\lambda_{\pm n}$ *is differentiable and*

$$\lambda'_{\pm n}(\beta) = \pm\{|u_{\pm n}|^2\,(b_0) + |pu'_{\pm n}|^2\,(b_0)\}. \tag{4.13}$$

(vii) *Assume that the boundary condition in* ω_0 *is coupled and is written in the canonical form* (2.9)*; in this case,* A_0, B_0 *in* ω_0 *are replaced by* θ_0, K_0. *Fix all components of* ω_0 *except* θ_0 *and consider* $\lambda_{\pm n}$ *as a function of* θ, *then* $\lambda_{\pm n}$ *is a differentiable function of* θ *and*

$$\lambda'_{\pm n}(\theta) = \pm[-2\,\mathrm{Im}(u_{\pm n}(b_0)(p\bar{u}'_{\pm n})(b_0))], \tag{4.14}$$

where Im z *denotes the imaginary part of* z. *Fix all components of* ω_0 *except* K_0 *and consider* $\lambda_{\pm n}$ *as a function of* $K \in SL(2, \mathbb{R})$. *Assume further that* $\lambda_{\pm n}(K_0)$ *is simple. Then,* $\lambda_{\pm n}(K)$ *is differentiable in some neighborhood* $\mathcal{N}_{K_0}$ *of* K_0 *in* $SL(2, \mathbb{R})$ *and for any* $K \in \mathcal{N}_{K_0}$,

$$\lambda'_{\pm n}(K)\,KH = \pm\left\{(p\bar{u}'_{\pm n}(b_0) \quad -\bar{u}_{\pm n}(b_0))\,KH\begin{pmatrix} u(b_0) \\ (pu')(b_0)\end{pmatrix}\right\},$$
$$H \in \mathbb{R}^{2\times 2}, \quad \mathrm{tr}\,H = 0. \tag{4.15}$$

Proof. The proof of this theorem is similar to the proofs of the corresponding results for the RD case ([19–22]) and, therefore, is omitted. Here we just mention that for any $K \in \mathrm{SL}(2, \mathbb{R}) \subset \mathbb{R}^{2\times 2}$, the tangent space of $\mathrm{SL}(2, \mathbb{R})$ at K is $\{KH;\, H \in \mathbb{R}^{2\times 2}, \mathrm{tr}\,H = 0\} \subset \mathbb{R}^{2\times 2}$. ∎

The derivative formulas (4.7)–(4.13) imply the following comparison results on the eigenvalues $\{\lambda_n;\, n \in \mathbb{Z}^*\}$, in which D will denote the Dirichlet BC.

THEOREM 4.3. (i) *Assume that* $\omega_k = (a_k, b_k, 1/p, q, w, D) \in \mathbf{\Omega}$ *for* $k = 0, 1$ *satisfy*

$$a_0 \leqslant a_1, \qquad b_0 \geqslant b_1. \tag{4.16}$$

If ω_1 *is left-definite and right-indefinite, then so is* ω_0, *and*

$$\lambda_n(\omega_0) \leqslant \lambda_n(\omega_1), \qquad \lambda_{-n}(\omega_0) \geqslant \lambda_{-n}(\omega_1) \tag{4.17}$$

for any $n \in \mathbb{N}_0$.

(ii) *Assume that* $\omega_k = (a, b, 1/p_k, q_k, w, A, B) \in \mathbf{\Omega}$ *for* $k = 0, 1$ *satisfy*

$$1/p_0 \geqslant 1/p_1, \qquad q_0 \leqslant q_1 \tag{4.18}$$

a.e. on J, *or* $\omega_k = (a, b, 1/p, q, w, \alpha_k, \beta_k) \in \mathbf{\Omega}$ *for* $k = 0, 1$ *satisfy*

$$\pi > \alpha_0 \geqslant \alpha_1 \geqslant 0, \qquad 0 < \beta_0 \leqslant \beta_1 \leqslant \pi. \tag{4.19}$$

If $\boldsymbol{\omega}_0$ *is left-definite and right-indefinite, then so is* $\boldsymbol{\omega}_1$, *and* (4.17) *holds for any* $n \in \mathbb{N}_0$.

(iii) *Let* $\boldsymbol{\omega}_k = (a, b, 1/p, q, w_k, A, B) \in \boldsymbol{\Omega}$ *be right-indefinite for* $k = 0, 1$. *If one of* $\boldsymbol{\omega}_0$ *and* $\boldsymbol{\omega}_1$ *is left-definite, then so is the other one. Moreover, in this case, if we also have*

$$w_0 \geqslant w_1 \tag{4.20}$$

a.e. on J, *then*

$$\lambda_n(\boldsymbol{\omega}_0) \leqslant \lambda_n(\boldsymbol{\omega}_1) \tag{4.21}$$

for any $n \in \mathbb{Z}^*$.

Proof. The proofs are similar to those of the corresponding results (Theorem 4.1 in [21]) in the RD case and, hence, are omitted. ∎

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